Recursion Examples

Recursion (review)

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Scenario: You are waiting in line for a concert. You can't see the front of the line, but you want to know your place in the line.

The person at the front, knows they are at the front!	Base case
You ask the person in front of you: “what is your place in the line?”	Recursive call
When the person in front of you figures it out and tells you, add one to that answer.	Use the solution to the smaller problem

Iteration vs. Recursion

- Iteration and recursion are somewhat related
- Converting **iteration to recursion** is formulaic, but converting **recursion to iteration** can be more tricky

Iterative

```
def fact_iter(n):
    total, k = 1, 1
    while k <= n:
        total, k = total*k, k+1
    return total
```

$$n! = \prod_{k=1}^{n} k$$

Names: n, total, k, fact_iter

Recursive

```
def fact(n):
    if n == 0:
        return 1
    else:
        return n * fact(n-1)
```

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

Names: n, fact

Sum Digits

Let's implement a recursive function to sum all the digits of `n`. Assume `n` is positive.

```
def sum_digits(n):
    """Calculates the sum of the digits `n`.
    >>> sum_digits(8)
    8
    >>> sum_digits(18)
    9
    >>> sum_digits(2018)
    11
    """
    "*** YOUR CODE HERE ***"
```

1. One or more **base cases**
2. One or more **recursive calls** with simpler arguments.
3. **Using the recursive call** to solve our larger problem.

Sum Digits

```python
def sum_digits(n):
    """Calculates the sum of the digits n
    >>> sum_digits(8)
    8
    >>> sum_digits(18)
    9
    >>> sum_digits(2018)
    11
    """
    if n < 10:
        return n
    else:
        all_but_last, last = n // 10, n % 10
        return sum_digits(all_but_last) + last
```

Order of Recursive Calls

Demo

Cascade

Goal: Print out a cascading tree of a positive integer `n`.

```
>>> cascade(486)
486
48
4
48
486
>>> cascade(48)
48
4
48
>>> cascade(4)
4
```

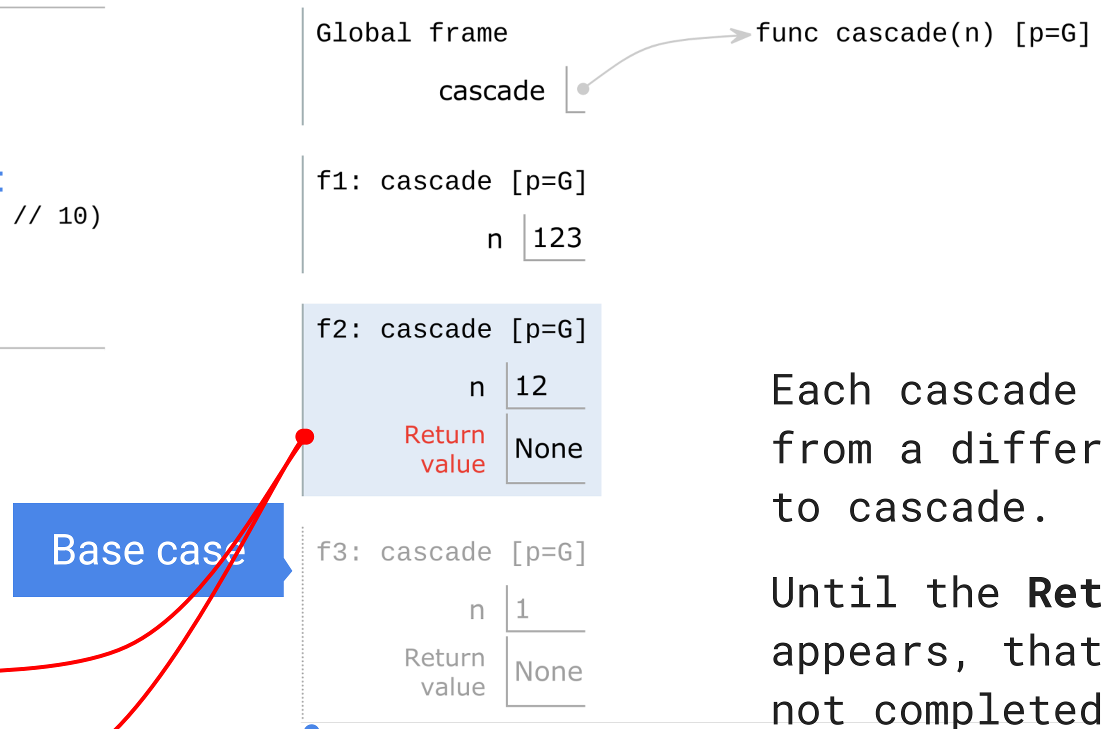
Ideas:

- If `n` is a single digit, just print it out!
- Otherwise, let `cascade(n // 10)` take care of the middle and `print(n)` around it

```python
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n // 10)
        print(n)
```

The Cascade Function

```python
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n // 10)
        print(n)

cascade(123)
```



Output

123
12
1
12

Each cascade frame is from a different call to cascade.

Until the **Return value** appears, that call has not completed.

Any statement can appear **before** or **after** the recursive call.

Demo

Two Implementations of Cascade

```python
def cascade(n):
  if n < 10:
      print(n)
  else:
      print(n)
      cascade(n // 10)
      print(n)
```

```python
def cascade(n):
  print(n)
  if n >= 10:
      cascade(n // 10)
      print(n)
```

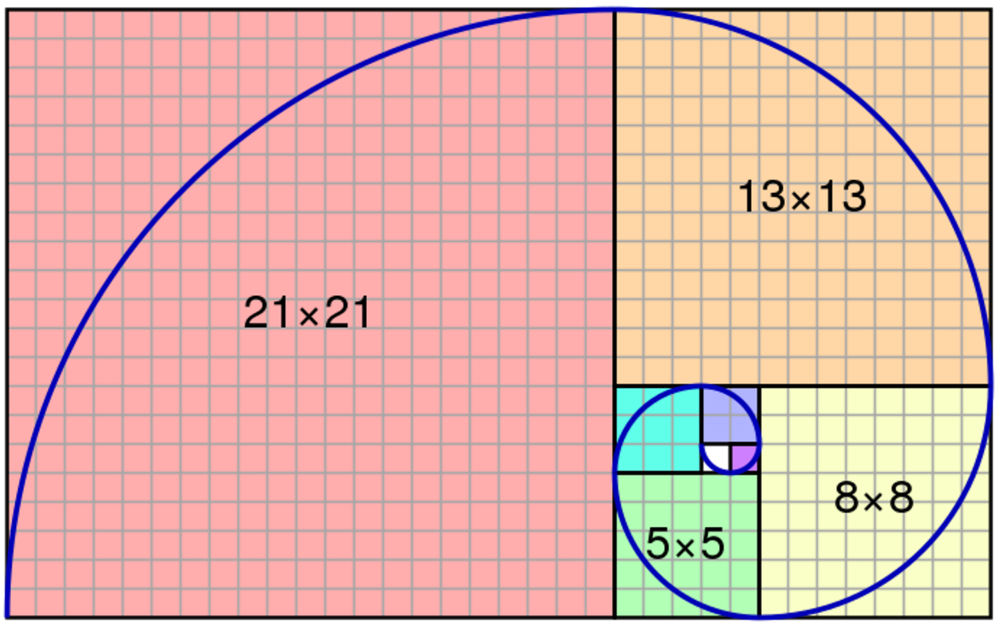
- If two implementations are equally clear, then shorter is usually better
- In this case, the longer implementation might be more clear
- When learning to write recursive functions, put the base case/s first

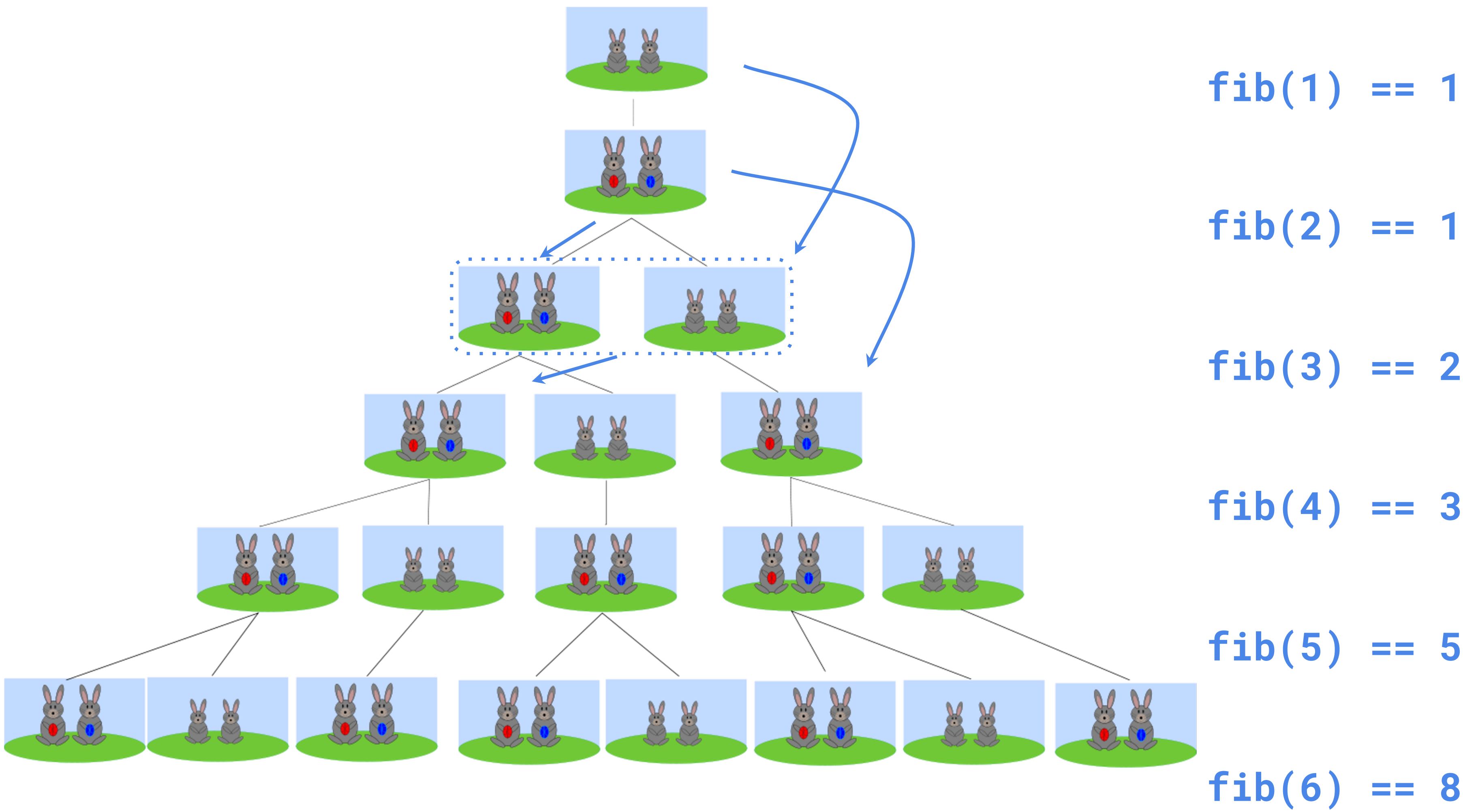
Fibonacci

Fibonacci Sequence

n	0	1	2	3	4	5	6	7	8	...	30
fib(n)	0	1	1	2	3	5	8	13	21	...	832040

+ + + + + + +





Fibonacci's rabbits

Demo

Fibonacci

Goal: Return the n th Fibonacci number.

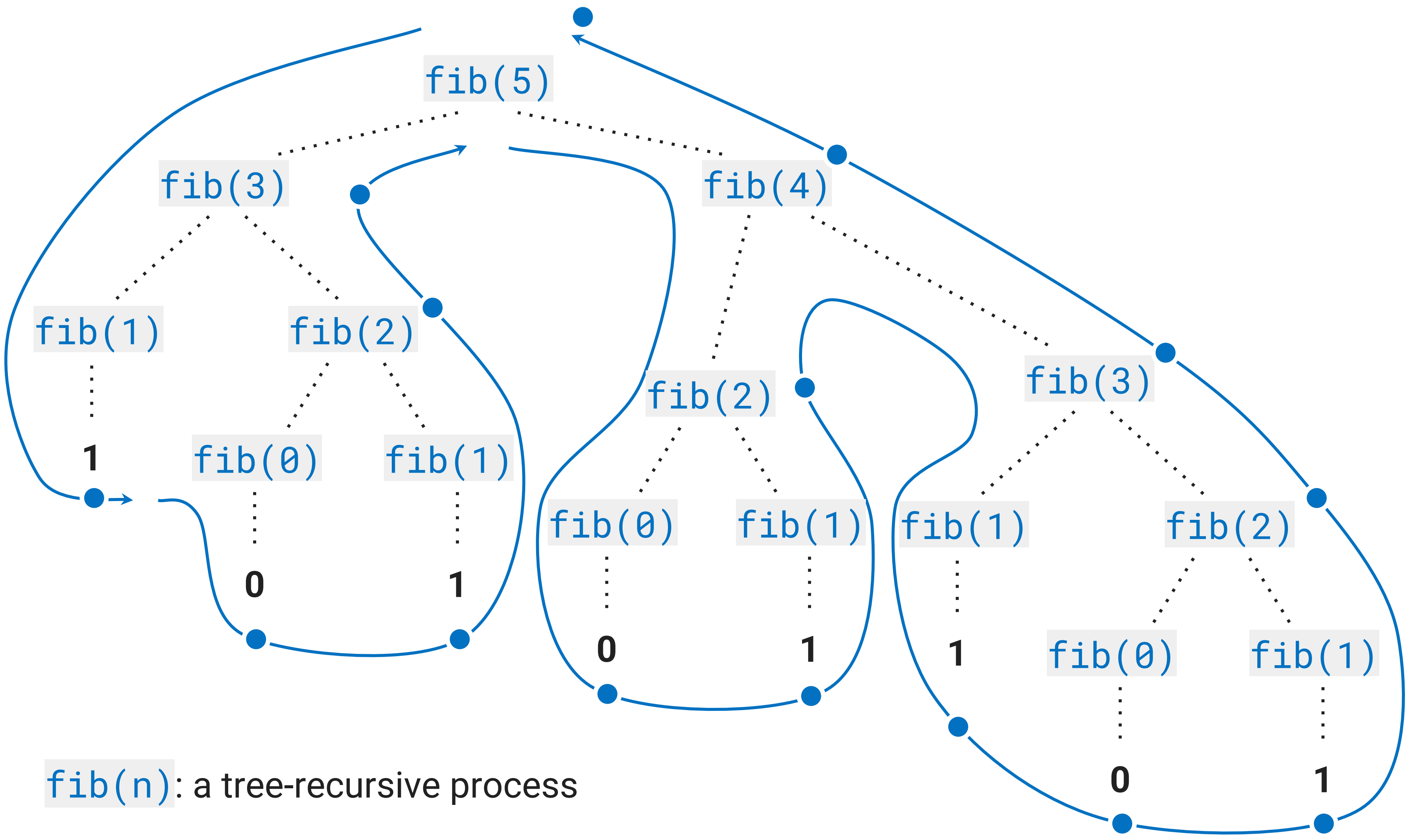
n	0	1	2	3	4	5	6	7	8	...	30
fib(n)	0	1	1	2	3	5	8	13	21	...	832040

Ideas:

- The first two Fibonacci numbers are known; if we ask for the 0th or 1st Fibonacci number, we know it immediately
- Otherwise, we sum up the previous two Fibonacci numbers

```
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n - 2) + fib(n - 1)
```

Fibonacci Call Tree



`fib(n)`: a tree-recursive process

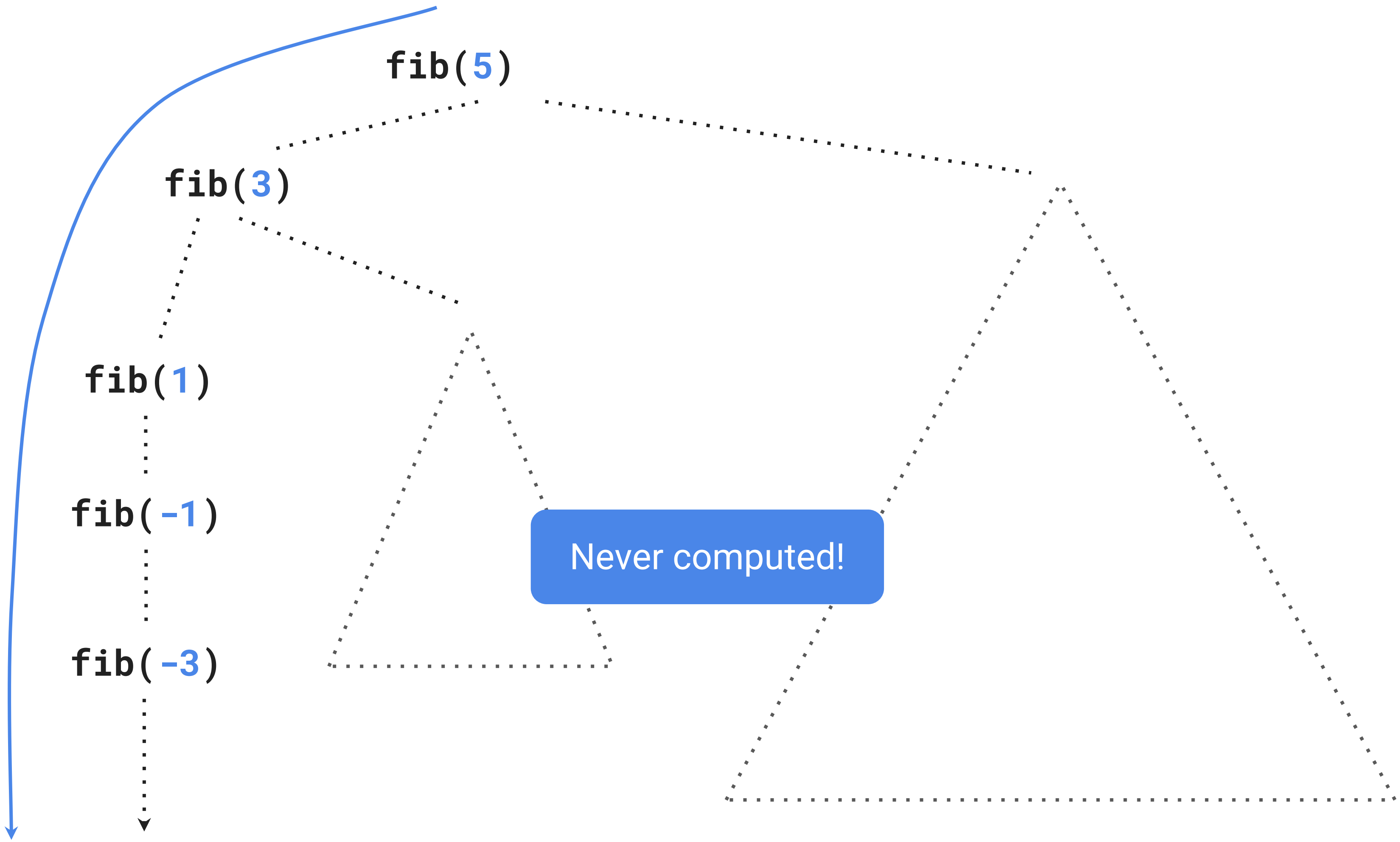
Broken Fibonacci

a. Wrong value
b. Error

```python
def broken_fib(n):
        if n == 0:
            return 0
        # Missing base case!
        else:
            return broken_fib(n - 2) +
                   broken_fib(n - 1)
```

```
>>> broken_fib(5)
Traceback (most recent call last):
...
RecursionError: maximum recursion
depth exceeded in comparison
```

```
broken_fib(5)
     ↓
broken_fib(3)
     ↓
broken_fib(1)
     ↓
broken_fib(-1)
     ⋮
```



Broken fib(n)

Counting Partitions

Count Partitions

Goal: Count the number of ways to give out `n` (> 0) pieces of chocolate if nobody can have more than `m` (> 0) pieces.

"How many different ways can I give out 6 pieces of chocolate if nobody can have more than 4 pieces?"

```
>>> count_part(6, 4)
9
```

Largest Piece: 4

2 + 4 = 6
1 + 1 + 4 = 6

Largest Piece: 3

3 + 3 = 6
1 + 2 + 3 = 6
1 + 1 + 1 + 3 = 6

Largest Piece: 2

2 + 2 + 2 = 6
1 + 1 + 2 + 2 = 6
1 + 1 + 1 + 1 + 2 = 6

Largest Piece: 1

1 + 1 + 1 + 1 + 1 + 1 = 6

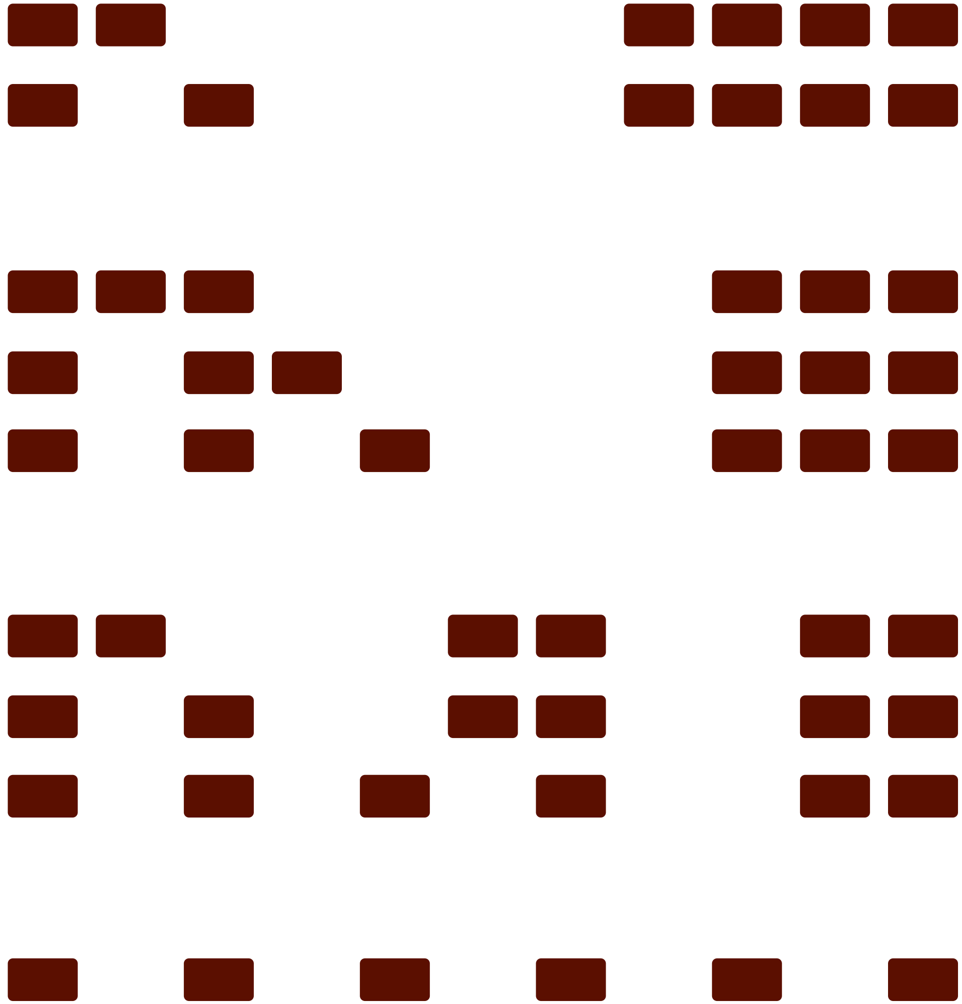
Count Partitions

2 + 4
1 + 1 + 4

3 + 3
1 + 2 + 3
1 + 1 + 1 + 3

2 + 2 + 2
1 + 1 + 2 + 2
1 + 1 + 1 + 1 + 2

1 + 1 + 1 + 1 + 1 + 1



Count Partitions

<u>Ideas</u>:

Find simpler instances of the problem

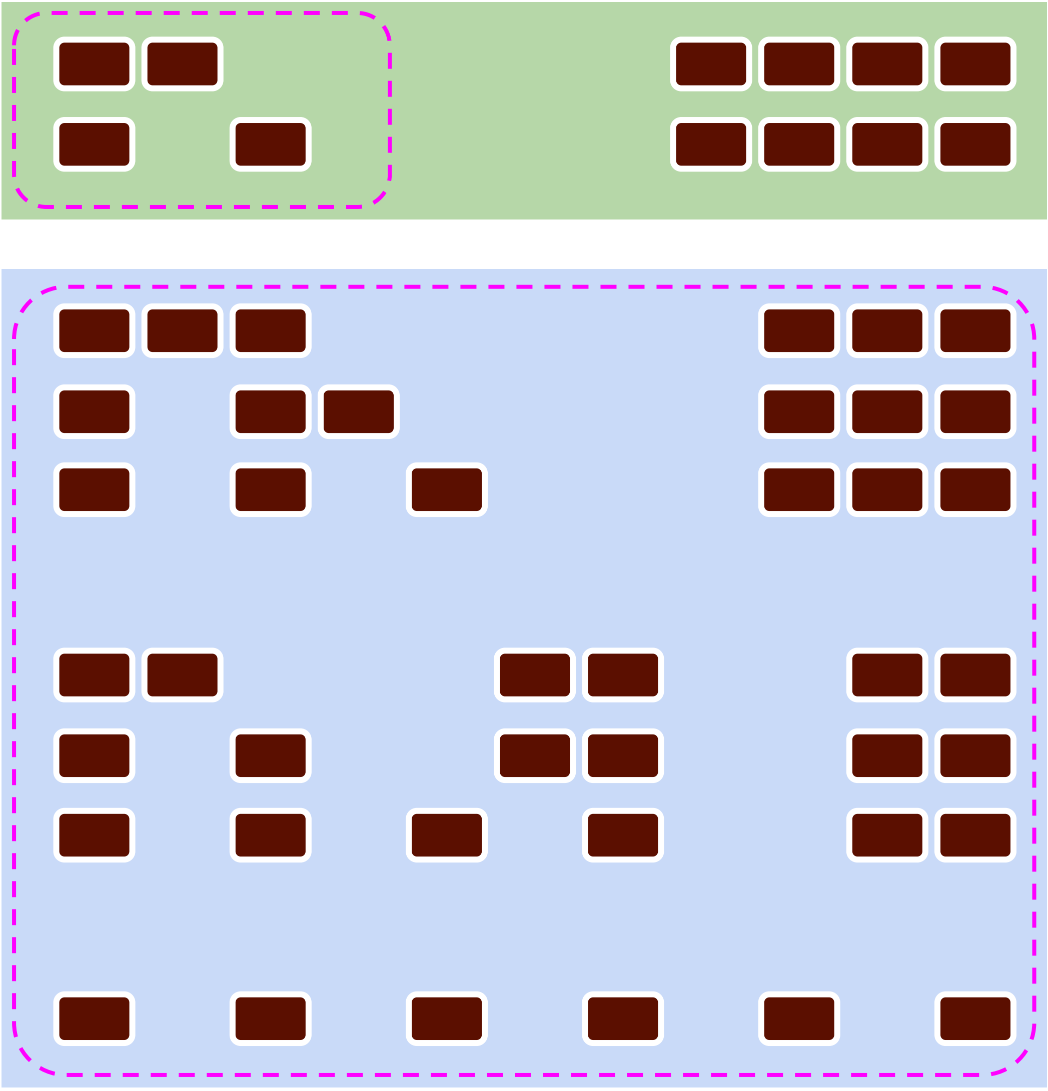
Explore two possibilities:

- Use a 4
- Don't use a 4

Solve two simpler problems:

- `count_part(2, 4)`
- `count_part(6, 3)`

Sum up the results of these smaller problems!



Count Partitions

<u>Ideas</u>:

Find simpler instances of the problem

Explore two possibilities:

- Use a 4
- Don’t use a 4

Solve two simpler problems:

- `count_part(2, 4)`
- `count_part(6, 3)`

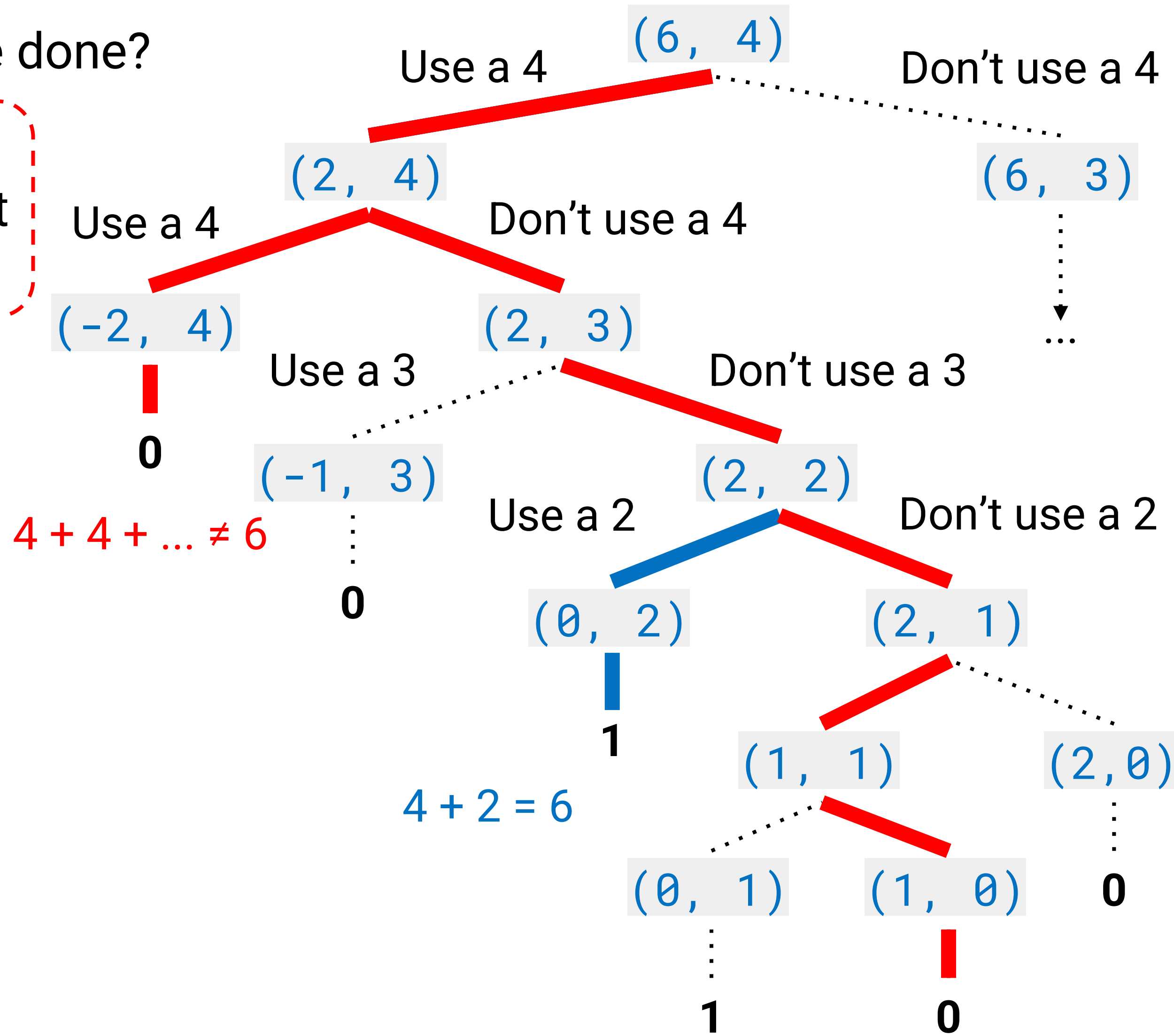
Sum up the results of these smaller problems!

```
def count_part(n, m):
      if




      else:
        with_m = count_part(n-m, m)
        wo_m = count_part(n, m - 1)
        return with_m + wo_m
```

Count Partitions

How do we know we're done?

- If `n` is negative, then we cannot get to a valid partition
- If `n` is 0, then we have arrived at a valid partition
- If the largest piece we can use is 0, then we cannot get to a valid partition



Demo

Count Partitions

<u>Ideas</u>:

Explore two possibilities:

- Use a 4
- Don't use a 4

Solve two simpler problems:

- `count_part(2, 4)`
- `count_part(6, 3)`

Sum up the results of these smaller problems!

How do we know we're done?

- If `n` is 0, then we have arrived at a valid partition
- If `n` is negative, then we cannot get to a valid partition
- If the largest piece we can use is 0, then we cannot get to a valid partition

```python
def count_part(n, m):
    if n == 0:
        return 1
    elif n < 0:
        return 0
    elif m == 0:
        return 0
    else:
            with_m = count_part(n-m, m)
            wo_m = count_part(n, m - 1)
            return with_m + wo_m
```

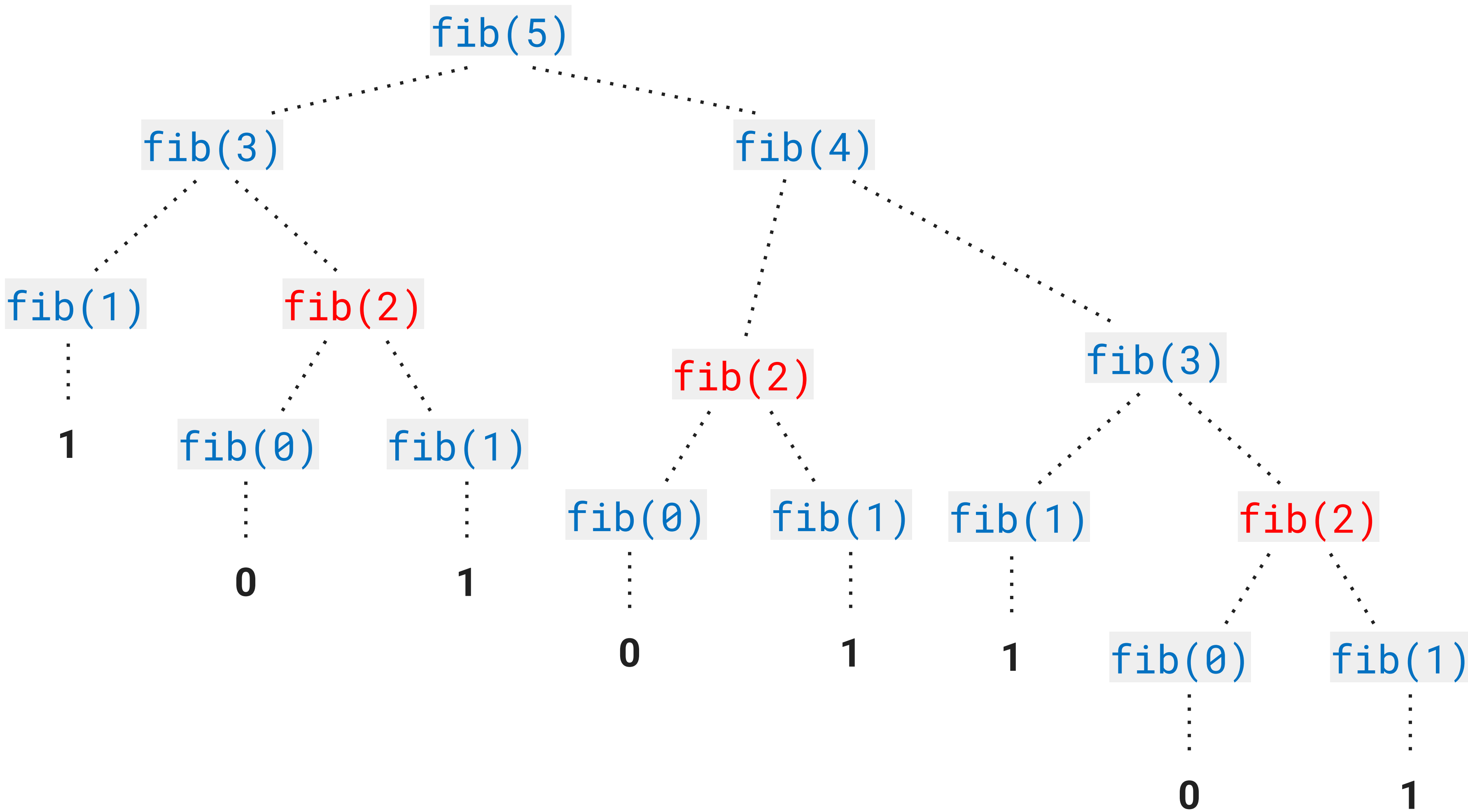
Takeaways

- Tree recursion allows you to **explore different possibilities**
- Oftentimes, the recursive calls for tree recursion represent different choices
    - One such choice is “do I use this value, or do I try another?”
- Sometimes it is easier to start with the recursive cases, and see which base cases those lead you to

If Time - Speeding Up Recursion (Teaser for the ~Future~)

Back to Fib

Demo



Basic Idea to Improve:

```
def better_fib(n):
        if n == 0:
            return 0
        elif n == 1:
            return 1
        elif already called better_fib(n):
            return stored value
        else:
            store & return better_fib(n - 2) + better_fib(n - 1)
```

Summary

- **Recursion** has three main components
  - **Base case/s**: The simplest form of the problem
  - **Recursive call/s**: Smaller version of the problem
  - Use the solution to the smaller version of the problem to arrive at the solution to the original problem
- When working with recursion, use *functional abstraction*: assume the recursive call gives the correct result
- **Tree recursion** makes multiple recursive calls and explores different choices
- Use doctests and your own examples to help you figure out the simplest forms and how to make the problem smaller